Entropic characterization of separability in Gaussian states

Sudha,[1, *] A. R. Usha Devi,[2, 3] and A. K. Rajagopal[3]

[1]*Department of Physics, Kuvempu University, Shankaraghatta, Shimoga-577 451, India*
[2]*Department of Physics, Bangalore University, Bangalore-560 056, India*
[3]*Inspire Institute Inc., McLean, VA 22101, USA.*

(Dated: February 28, 2010)

We explore separability of bipartite divisions of mixed Gaussian states based on the positivity of the Abe-Rajagopal (AR) q-conditional entropy. The AR q-conditional entropic characterization provide more stringent restrictions on separability (in the limit $q \to \infty$) than that obtained from the corresponding von Neumann conditional entropy ($q = 1$ case) – similar to the situation in finite dimensional states. Effectiveness of this approach, in relation to the results obtained by partial transpose criterion, is explicitly analyzed in three illustrative examples of two-mode Gaussian states of physical significance.

PACS numbers: 03.67.Mn, 03.65.Ud

Characterizing separability of a multipartite quantum state is a central issue in the subject of quantum information. Given density matrix of a composite system, it is hard to decide its separability status, solely based on its intrinsic properties. In 1989 Werner [1] defined *inseparability* by pointing out the impossibility of expressing an entangled composite quantum state as a convex mixture of its subsystem states. Peres [2] enunciated *positivity under partial transpose* (PPT) criterion for separability of bipartite states based on this definition in 1996. The PPT criterion was soon shown to be both necessary and sufficient in finite dimensional 2×2 and 2×3 systems by R. Horodecki [3]. Peres' criterion has also led to an often-used quantifying measure of entanglement viz., negativity/logarithmic negativity [4]. Much of the work that followed ever since has been focused on identifying less formidable sufficient – though not necessary – conditions for separability, as well as other mathematical methods for their analysis, such as positive and completely positive maps. A comprehensive review of these works on finite dimensional discrete systems and less extensively on the continuous systems may be found in the recent review article by Horodecki et. al. [5].

Besides finite dimensional discrete systems, the issue of separability in continuous variable (CV) composite states, such as coupled bosonic oscillator systems (light modes), belonging to infinite dimensional Hilbert spaces, too has invited much attention. The importance of investigating CV systems is evidenced by the tremendous activity in this field, as is clear from the review articles on this topic [6]. Fortunately, Peres' criterion has been extended to bipartite CV states and is found to be both necessary and sufficient for two-mode Gaussian states [7, 8]. In fact, Gaussian states form a distinguished class amongst the CV systems due to experimental and theoretical ease they offer. Logarithmic negativity [4] has also been employed to quantify entanglement in mul-

timode Gaussian states and it provides a necessary and sufficient way of characterizing entanglement in the case of two mode Gaussian states. Quantification of entanglement of two-mode Gaussian states in terms of minimal set of local measurements and classical communication has been developed in Ref. [9]. Further, entanglement of formation has been analytically computed for arbitrary two mode Gaussian states [10].

A physically elegant method to characterize separability is based on the use of global and local spectra of the composite quantum system – which forms the basis of entropic approach for separability ([11]– [16]). Whereas the non-negativity of von-Neumann conditional entropy is used to identify entangled pure states, it is inadequate to address the issue of separability in mixed states. Generalized entropic measures ([11] –[16]) offer more sophisticated tools to explore global vs local disorder in mixed states and lead to stringent limitation on separability than that obtained using positivity of von Neumann conditional entropy. Horodecki et. al. [12] recognized that conditional Renyi entropies are necessarily non-negative for all separable states, while they can assume negative values by entangled states. Employing Tsallis entropy [13], indexed by a real parameter $q \in [0, \infty]$, Abe and Rajagopal [14] defined q-conditional entropy associated with the bipartite division of a density matrix $\rho(A, B)$ and its subsystem $\rho(A) = \mathrm{Tr}_B [\rho(A, B)]$ as

$$
\begin{aligned}
S_q(B|A) &= \frac{1}{1-q} \left[1 - \frac{\mathrm{Tr}(\rho^q(A, B))}{\mathrm{Tr}(\rho^q(A))} \right] \\
&= \frac{1}{1-q} \left[1 - \frac{\sum_n \lambda_n^q(A, B)}{\sum_m \lambda_m^q(A)} \right]
\end{aligned}
\tag{1}
$$

(where $\lambda_n(A, B)$, $\lambda_m(A)$ are the eigenvalues of $\rho(A, B)$ and $\rho(A)$ respectively [17]) and employed it to investigate the issue of separability. Tsallis q-conditional entropy method (AR approach) has also been employed to investigate separability in several finite dimesional quantum systems [15]. As any spectral criteria, based only on the eigenvalues of the state and its subsystems, do not provide a complete characterization of separability [16], the

AR q-conditional entropy characterization does not lead, in general, to the necessary and sufficient criteria for separability. However, this approach is fruitful in obtaining stronger criteria than the one derived from the familiar $q = 1$ case [17] i.e, the result based on von Neumann conditional entropy.

The AR q-entropy approach relies on finding the global and local spectra of the density matrices, which are not straightforward in the case of CV systems. However, for n-mode Gaussian states, one can evaluate finite number (n) of symplectic eigenvalues [18] of the corresponding $2n \times 2n$ variance matrix (which completely characterizes the Gaussian state) – in terms of which the eigenvalues of the density matrix may be expressed readily [19–21]. However, the issue of separability based on conditional q-entropy approach has not been addressed so far in the context of Gaussian states, to the best of our knowledge. The present Brief Report aims towards investigating separability in Gaussian states based on AR q-entropic approch, thus filling an important gap.

We consider n-mode Gaussian states, which are completely determined by the $2n \times 2n$ covariance matrix $V_{\alpha\beta} = \frac{1}{2}\langle\{\triangle\xi_\alpha, \triangle\xi_\beta\}\rangle$, $\alpha,\beta = 1, 2, \ldots, 2n$; $\triangle\xi = \xi - \langle\xi\rangle$, $\{O_1, O_2\} = O_1 O_2 + O_2 O_1$ and $\langle O\rangle = \text{Tr}[\rho O]$ denotes the expectation value of the operator O. Under a $2n \times 2n$ symplectic transformation [18] $S \in \text{Sp}(2n, \text{R})$, a Gaussian state is mapped to another Gaussian state characterized by the covariance matrix $V' = SVS^T$. Then, it follows from Williamson theorem that for every covariance matrix V there exists a symplectic matrix S such that $SVS^T = \text{diag}(\nu_1, \nu_1; \nu_2, \nu_2; \ldots; \nu_n, \nu_n)$, where $\nu_k, k = 0, 1, \ldots, n$ denote the symplectic eigenvalues [18]. Correspondingly, the associated density matrix is expressed as a tensor product of n thermal states of oscillators:

$$\rho_n \rightarrow \rho'_n = U(S)\,\rho_n\,U^\dagger(S) = \bigotimes_{k=1}^{n} \rho(\nu_k) \qquad (2)$$

where $\rho(\nu_k) = \frac{1}{\nu_k+\frac{1}{2}}\sum_{j=0}^{\infty}\left(\frac{\nu_k-\frac{1}{2}}{\nu_k+\frac{1}{2}}\right)^j |j\rangle_k\langle j|$. (Here $\{|j\rangle_k, \; j = 0, 1, \ldots, \infty\}$ denote the number states of the kth mode). An arbitrary positive power $\text{Tr}[\rho^q]$, $0 < q \leq \infty$ of the n-mode Gaussian density operator may thus be readily expressed in terms of the symplectic eigenvalues as [21],

$$\text{Tr}[\rho_n^q] = \prod_{k=1}^{n} \text{Tr}\,[\rho^q(\nu_k)] = \prod_{k=1}^{n} \frac{1}{(\nu_k + \frac{1}{2})^q - (\nu_k - \frac{1}{2})^q}.$$

Considering a bipartite division of a n mode Gaussian system $\rho_n(A, B)$, with marginals $\text{Tr}_B[\rho_n(A, B)] = \rho_N(A)$, $\text{Tr}_A[\rho_n(A, B)] = \rho_{(n-N)}(B)$ (where $A \rightarrow N$ modes, $B \rightarrow (n - N)$ modes, $N < n$), the AR q-conditional entropy Eq. (1) associated with Gaussian states is readily expressible in terms of respective symplectic eigenvalues $\nu_k^{(AB)}$, $\nu_l^{(A)}$ of $\rho_n(A, B)$ and $\rho_N(A)$

as

$$S_q(B|A) = \frac{1}{q-1}\left[1 - \frac{\prod_{l=1}^{N}\left[\left(\nu_l^{(A)}+\frac{1}{2}\right)^q - \left(\nu_l^{(A)}-\frac{1}{2}\right)^q\right]}{\prod_{k=1}^{n}\left[\left(\nu_k^{(AB)}+\frac{1}{2}\right)^q - \left(\nu_k^{(AB)}-\frac{1}{2}\right)^q\right]}\right] \qquad (3)$$

The q-conditional entropy is necessarily positive, when the modes A, B are separable. Negative values of $S_q(B|A)$ therefore imply entanglement between the modes A and B – offering a sufficient condition to characterize entanglement in Gaussian states [7, 8].

On the other hand the PPT criterion translates itself to the following constraint: the lowest symplectic eigenvalue $\tilde{\nu}_{\min}$ of the variance matrix $\tilde{V}$ (where the canonical momenta p_l of the transposed modes reverse their sign [7]) of the partially transposed density matrix ρ^T satisfies $\tilde{\nu}_{\min} \geq \frac{1}{2}$ for all separable Gaussian states [4, 7]. Violation of this condition viz., $\tilde{\nu}_{\min} < \frac{1}{2}$ is a characteristic of entanglement. This PPT based characterization serves as a necessary and sufficient condition for separability in two mode Gaussian states.

To examine the utility of the AR q-entropy approach, we will discuss separability of mixed two-mode Gaussian states of physical importance. We compare the inseparability range obtained using the q-entropy criteria with that obtained using conditional von-Neumann entropy and also that resulting from PPT.

Two mode squeezed thermal state: Density matrix of the two mode squeezed thermal state is given by [22]

$$\rho(A, B) = U(S_r)\,\rho_{\text{th}}(A) \otimes \rho_{\text{th}}(B)\,U^\dagger(S_r).$$

Here $U(S_r) = \exp\left[\frac{r}{2}(a_1^\dagger a_2^\dagger - a_1 a_2)\right]$ corresponds to the two-mode squeezing operator [23]; r is the real positive squeezing parameter and $\rho_{\text{th}}(A)$, $\rho_{\text{th}}(B)$ denote single mode thermal states, both at same temperature T.

The variance matrix $V(A, B)$ of the two-mode squeezed thermal state is given explicitly by

$$V(A, B) = \frac{\coth(\beta/2)}{2}\begin{pmatrix} \cosh r & 0 & \sinh r & 0 \\ 0 & \cosh r & 0 & -\sinh r \\ \sinh r & 0 & \cosh r & 0 \\ 0 & -\sinh r & 0 & \cosh r \end{pmatrix} \qquad (4)$$

where $\beta = T^{-1}$ is the inverse temperature (which is a dimensionless parameter with the choice of appropriate units i.e., $\hbar$, the oscillator frequency ω and the Boltzmann constant κ are equal to one). The symplectic eigenvalues $\nu_k^{(AB)}$, $k = 1, 2$ associated with this state are degenerate and are given by $\nu_{k=1,2}^{(AB)} = \frac{\coth(\beta/2)}{2}$. The symplectic eigenvalue of the reduced density matrix $\rho(A)$ is found to be $\nu(A) = \frac{\coth(\beta/2)\,\cosh r}{2}$. Now, using Eq. (3), the conditional q-entropy associated with two mode squeezed thermal state may be readily obtained as

$$S_q(B|A) = \frac{1}{q-1}\left[\frac{(\coth(\beta/2)\,\cosh r + 1)^q - (\coth(\beta/2)\,\cosh r - 1)^q}{[(\coth(\beta/2) + 1)^q - (\coth(\beta/2) - 1)^q]^2}\right] \qquad (5)$$

An implicit plot of $S_q(B|A) = 0$ (see Fig. 1) shows that $T_c^{(\infty)} \rightarrow 2.82$ in the limit $q \rightarrow \infty$, for $r = 2$. One can also
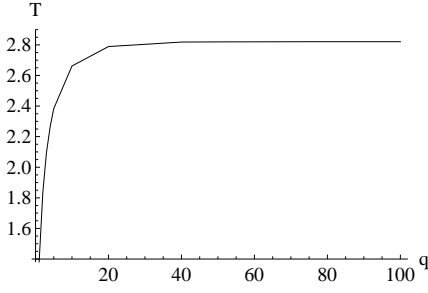
FIG. 1: Implicit plot of $S_q(B|A) = 0$ (with the choice of the parameter $r = 2$) as function of q for the two-mode squeezed thermal state.

FIG. 2: The conditional q-entropy $S_q(B|A)$ for different values of q, as a function of temperature $T = \beta^{-1}$ of the Gaussian state resulting by combining a single mode squeezed state (with squeezing parameter $\eta = 4$) with a thermal state in a $50:50$ beam splitter.

see that the temperature $T_c^{(1)} \approx 1.381$ above which the conditional von-Neumann entropy $S_1(B|A)$ is positive. It is clear that the threshold temperature T_c increases with increasing q and the strongest limitation on separability results when $q \to \infty$.

In order to compare the effectiveness of the AR q-entropic characterization with that based on the PPT criterion, we identify that the minimum symplectic eigenvalue of the partially transposed squeezed thermal state $\tilde{\nu}_{\min} = \frac{1}{2} e^{-r} \coth \frac{\beta}{2}$ is less than $\frac{1}{2}$ when $T_c^{\mathrm{PPT}} \geq 3.672$, for $r = 2$. This clearly reveals that the separability domains inferred via the threshold temparature values follow the trend $T_c^{(1)} < T_c^{(\infty)} < T_c^{\mathrm{PPT}}$. In other words, the PPT criterion gives the strongest limitation [24] (which is both necessary and sufficient) on separability.

Two-mode state resulted by combining a squeezed state and a thermal state in a 50:50 beam splitter: Now we consider a two mode Gaussian state obtained when a single mode squeezed state interferes with a single mode thermal state through a $50:50$ beam splitter [19]. The variance matrix of the resulting two-mode state is given by [19]

$$V(A,B) = \frac{1}{4} \begin{pmatrix} a+b & 0 & a-b & 0 \\ 0 & a+\frac{1}{b} & 0 & a-\frac{1}{b} \\ a-b & 0 & a+b & 0 \\ 0 & a-\frac{1}{b} & 0 & a+\frac{1}{b} \end{pmatrix} \quad (6)$$

Here $b = e^\eta$ with η denoting the single mode squeezing parameter and $a = \coth(\beta/2)$, where $\beta = T^{-1}$ corresponds to the inverse temperature of the input thermal state.

The symplectic eigenvalues of $V(A,B)$ are non-degenerate and are found to be

$$\nu_1^{(AB)} = \frac{1}{2}; \quad \nu_2^{(AB)} = \frac{1}{2} \coth \frac{\beta}{2}. \quad (7)$$

The symplectic eigenvalue of $V(A)$ (and also $V(B)$) is found to be

$$\nu^{(A)} = \frac{1}{4}\sqrt{(a+b)(a+\frac{1}{b})} = \frac{1}{4}\sqrt{1 + 2\cosh\eta \coth\frac{\beta}{2} + \coth^2\frac{\beta}{2}}. \quad (8)$$

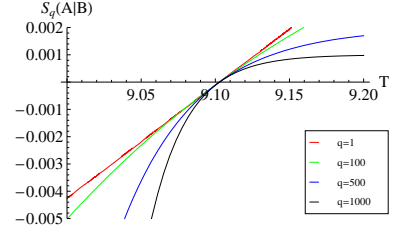Substituting Eqs. (7) and (8) in (3) one can obtain an explicit expression for $S_q(B|A)$.

In Fig. 2 we illustrate the variation of $S_q(B|A)$ for different choices of the parameter q. It is evident from Fig. 2 that the AR q-entropy ceases to be negative at $T_c \approx 9.1$ for different choices of q. In other words, both conditional von-Neumann entropy $S_1(B|A)$ and the AR q entropy, in the limit $q \to \infty$, lead to the same inseparability range. The exact inseparability range obtained by PPT criterion is much stronger (for the choices of the parameters we find that the condition $\tilde{\nu}_{\min} = 1/2$ on the lowest eigenvalue of $\tilde{V}(A,B)$ is satisfied for $T_c^{\mathrm{PPT}} \approx 27.3$). Thus we find that $T_c^{(1)} = T_c^{(\infty)} < T_c^{\mathrm{PPT}}$ for the state under consideration.

Two mode Squeezed state subjected to a coupled leaky wave guide: As our third example, we consider a mixed two-mode Gaussian state that results when a pure two-mode squeezed state is transmitted via a coupled waveguide system with non-zero leakage [25].

Transmission of a pure two mode squeezed vacuum state $\exp\left[\frac{r}{2}\left(a_1^\dagger a_2^\dagger - a_1 a_2\right)\right]|0,0\rangle$ via two leaky waveguides coupled to each other by an interaction term $H_{\mathrm{int}} = \mathcal{J}\left(a_1^\dagger a_2 + a_2^\dagger a_1\right)$, ($\mathcal{J}$ denotes the coupling strength), results in a mixed two mode Gaussian state, the variance matrix $V(A,B)$ of which is given by [25]

$$V(A,B) = \begin{pmatrix} f & g & h & 0 \\ g & f & 0 & -h \\ h & 0 & f & g \\ 0 & -h & g & f \end{pmatrix} \quad (9)$$

where $f = \frac{1}{2} + e^{-2\gamma t}\sinh^2(\frac{r}{2})$, $g = -\frac{1}{2}e^{-2\gamma t}\sinh r \sin(2\mathcal{J}t)$ and $h = \frac{1}{2}e^{-2\gamma t}\sinh r \cos(2\mathcal{J}t)$.

Here r denotes the squeezing parameter of the input state and γ corresponds to leakage (decay rate) of individual modes. Whereas the global symplectic spectra of $V(A,B)$ are given by $\nu_1^{(AB)} = \nu_2^{(AB)} = \sqrt{f^2 - g^2 - h^2}$, its local symplectic eigenvalue is $\nu^{(A)} = \sqrt{f^2 - g^2}$. In fact, the time evolution of entanglement of the two-mode Gaussian state transmitted via a coupled leaky wave guide exhibits a damped oscillation pattern [25], and this oscillatory behavior repeats until a total decay takes place due to environmental decoherence. In the present
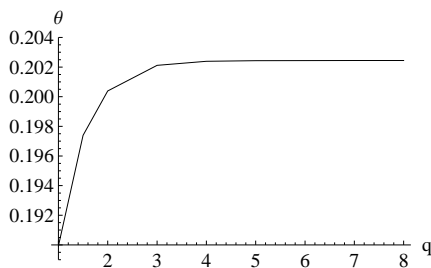
FIG. 3: An implicit plot of $S_q(B|A) = 0$ as a function of q for the two-mode squeezed state subjected to a coupled leaky wave guide. (Here, $r = 1.8$ and $\gamma/\mathcal{J} = 0.1$.)

discussion we have focussed only on the time duration for which the initially entangled two mode squeezed state loses its entanglement at a first glance during evolution. We find that the scaled time $\theta = \mathcal{J}t/\pi$ for which the conditional von-Neumann entropy $S_1(B|A)$ ceases to be negative is given by $\theta_c^{(1)} \approx 0.19$, when $r = 1.8$ and $\gamma/\mathcal{J} = 0.1$. Stronger limitations on separability follow with the increase of the parameter q. It is evident from

Fig. 3 that the largest scaled time interval $\theta_c^{(q)}$ (which approaches the value 0.202), after which the initially entangled two mode state becomes separable, is realized in the limit $q \to \infty$. On the other hand the necessary and sufficient condition for separability (identified by the condition $\tilde{\nu}_{\min} \geq 1/2$ on the smallest symplectic eigenvalue $\tilde{\nu}_{\min}$) leads to threshold scaled time $\theta_c^{\mathrm{PPT}} \approx 0.23$. Thus, it follows that $\theta_c^{(1)} < \theta_c^{(\infty)} < \theta_c^{\mathrm{PPT}}$.

In conclusion, we have explored separability in Gaussian states based on the AR q-conditional entropy approch. This is facilitated by expressing the q-conditional entropy in terms of the symplectic eigenvalues of the state. We have analyzed the separability features of three different examples of two-mode Gaussian states using this entropic approach and compared the results with those obtained from conditional von-Neumann entropy ($q = 1$ limit of AR q-entropy) and with the PPT method. Strongest limitation on separability is realized in the limit $q \to \infty$, although the q-entropy approach leads to weaker domain of separability than the exact one obtained from PPT method.

[1] R. F. Werner, Phys. Rev. A **40**, 4277 (1989).
[2] A. Peres, Phys. Rev. Lett. **77**, 1413 (1996).
[3] R. Horodecki, Phys. Lett. A **232**, 333 (1997).
[4] G. Vidal and R. F. Werner, Phys. Rev. A **65**, 032314 (2002).
[5] R. Horodecki, P. Horodecki, M. Horodecki and K. Horodecki, Rev. Mod. Phys. **81**, 865 (2009).
[6] S. L. Braunstein and A. K. Pati (eds), Computation and Quantum Information, (Kluwer, Dodorecht (2002)); S. L. Braunstein and P. van Loock, Rev. Mod. Phys. **77**, 513 (2005); S. Mancini and S. Swerini, *Electronic Notes in Theoretical of Comp. Sci.*, **169**, 121 (2007); Xiang-Bin Wang, Phys. Rep. **448**, 1 (2007).
[7] R. Simon, Phys. Rev. Lett. **84**, 2726 (2000).
[8] L- M. Duan, G. Giedke, J. I. Cirac, and P. Zoller, Phys. Rev. Lett. **84**, 2722 (2000).
[9] L. F. Haruna, M. C. de Oliveira, G. Rigolin, Phys. Rev. Lett. **98**, 150501 (2007); Phys. Rev. Lett. **99**, 059902 (2007). Also see G. Rigolin, C. O. Escobar, Phys. Rev. A **69**, 012307 (2004), for two lower bounds on entanglement of formation for arbitrary mixed Gaussian states.
[10] P. Marian, T. A. Marian, Phys. Rev. Lett. **101**, 220403 (2008); J. Soloman Ivan and R. Simon, quant-ph arXiv:0808.1658.
[11] N. J. Cerf, C. Adami, Phys. Rev. Lett. **79**, 5194 (1997).
[12] R. Horodecki, P. Horodecki, and M. Horodecki, Phys. Lett. A **210**, 377 (1996); R. Horodecki and M. Horodecki, Phys. Rev. A **54**, 1838 (1996).
[13] C. Tsallis, J. Stat. Phys. **52**, 479 (1988); C. Tsallis, R. S. Mendes and A. R. Plastino, Physica A **261**, 534 (1998)
[14] S. Abe and A. K. Rajagopal, Physica A **289**, 157 (2001).
[15] C. Tsallis, S. Lloyd, and M. Baranger, Phys. Rev. A **63**, 042104 (2001); A. K. Rajagopal and R. W. Rendell, Phys. Rev. A **66**, 022104 (2002); S. Abe, Phys. Rev. A **65**, 052323 (2002); J. Batle, M. Casas, A. R. Plastino, and A. Plastino, Eur. Phys. J. B **35**, 391 (2003); R. Prabhu, A. R. Usha Devi and G. Padmanabha, **76**, 042337 (2007).
[16] K. G. H. Vollbrecht and M. M. Wolf, J. Math. Phys. **43**, 4299 (2002); M. A. Nielsen and J. Kempe, Phys. Rev. Lett. **86**, 5184 (2001); D. Chruscinski, A. Kossakowski and G. Sarbicki, Phys. Rev. A **80**, 042314 (2009).
[17] Note that in the limit $q \to 1$ the AR q-conditional entropy Eq. (1) reduces to von Neumann conditional entropy i.e., $\lim_{q \to 1} S_q(B|A) \to S(B|A) = S(A, B) - S(B)$.
[18] R. Simon, N. Mukunda and B. Dutta, Phys. Rev. A **49**, 1567 (1994); Arvind, B. Dutta, N. Mukunda, R. Simon, Phys. Rev. A **52**, 1609 (1995).
[19] G. Adesso, A. Serafini, and F. Illuminati, Phys. Rev. A **70**, 022318 (2004).
[20] J. Soloman Ivan and R. Simon, quant-ph arXiv:0812.2805.
[21] S. Pirandola and S. Lloyd, Phys. Rev. A **78**, 012331 (2008).
[22] Xiao-Yu Chen, Pei-Iiang Qiu, Phys. Lett. A **314**, 191 (2003).
[23] P. L. Knight and V. Buzeck, "Squeezed states: basic principles in Quantum Squeezing", Edited by P. D. Drummond and Z. Ficek (Springer, 2004).
[24] The values of the threshold temperatures $T_c^{(1)}$, $T_c^{(\infty)}$ and T_c^{PPT} depend on the squeezing parameter r and it may be noted that for a higher value of squeezing parameter $r = 5$, the threshold temperatures take the values $T_c^{(1)} \approx 27.3$, $T_c^{(\infty)} \approx 73.2$, while $T_c^{\mathrm{PPT}} \approx 74.2$. In other words, the separability range inferred by q-conditional entropy method (in the limit $q \to \infty$) approaches closer to the exact one based on PPT for larger values of the squeezing parameter.
[25] A. Rai, S. Das and G. S. Agarwal, arXiv e-print:0907.2432.